Series Complex-Potential Solution of Flow Around Arbitrary Airfoils

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An accurate, concise, and numerically efficient method has been developed to solve incompressible irrotational flow around arbitrary airfoils. The airfoil is transformed into a near-circular shape by an inverse Joukowski transformation. The flow around the transformed shape is then solved by adding a uniform flow, a circulation, and a series complex potential. The method gives almost exact results for a number of Karman-Trefftz airfoils with widely varying geometrical parameters and for the NACA 0012 wing section. For most of these airfoils, the method has been found to be generally more accurate than the Hess-Smith panel method, but at a fraction of the numerical labor.

Introduction

THE calculation of potential flow around airfoils has been subject for research for more than half a century now. Early efforts centered around using the conformal mapping approach. In this approach, the flow around the airfoil is transformed through a number of transformations into the flow around a circular cylinder for which solution is known. The exact flowfield in the physical plane (around the airfoil) is then obtained by transforming the flow around the cylinder backward, using the same transformations used earlier. Methods based on this approach are exact, but they are limited to airfoils of specific geometries, e.g., Joukowski airfoil.[1] Although these airfoils do not have too much practical significance, their exact solutions are quite valuable when used as test cases for methods dealing with airfoils of arbitrary shapes. Theodorsen,[2] however, was able to extend this approach to deal with arbitrary airfoils by using two major transformations. He used an inverse Joukowski transformation to transform the airfoil into a near-circular shape followed by a complicated transformation to obtain a true circle. The method, though it gives accurate results, is mathematically involved and is quite cumbersome.

The second approach to calculate potential flow around airfoils is based on the distribution of potential singularities, such as vortices, sources/sinks, and doublets, within the airfoil or on its surface. Distribution of vortices along the camberline or the chord simplifies the analysis substantially. This approach, which is known as the "thin-airfoil" theory, provides "reasonably" accurate results for thin airfoils with small camber at small angles of attack.[3] Thickness effects can be approximately accounted for using a source/sink distribution along the chord. This yields additional and equal contributions to the velocity on both sides of the airfoil.[4] Recently, Zedan and Abu-Abdou[5] improved the accuracy of this method by providing an accurate procedure to determine the vortex and source coefficients and by developing expressions for the aerodynamic parameters that include the thickness effects. Moreover, the extended method has the capability of handling profiles with moderate trailing-edge angles. The method produced aerodynamic coefficients with accuracy comparable to that of the Hess-Smith (HS) panel method[6,7] for Karman-Trefftz airfoils with thickness ratios τ up to 16%, trailing-edge angles β up to 14 deg, and camber ratios ε up to 10%, at an angle of attack $\alpha = 6$ deg. However, the accuracy of the predicted pressure distribution was not as "good" for thick airfoils.

The other type of singularity methods that are known as the panel methods utilizes surface distributions of either sources, vortices, doublets, or combinations thereof. These methods, which are quite involved, became very popular after the advent of high-speed computers, because of their accuracy and ability to handle arbitrary airfoils and even three-dimensional configurations. However, their use does not guarantee complete accuracy, since there are lots of pitfalls in their application that can affect their performance adversely. According to Ardonceau,[8] "the uniqueness of the type and distributions of singularities that satisfy the boundary conditions may require a lot of know-how to carry out accurate computations." Ardonceau showed that low-order surface vortex method gives a nonrealistic loop in the pressure distribution near the trailing edge when the airfoil trailing edge is thin, as in the case of the laminar five-digit NACA wing sections. Similar behavior is known to occur when using the low-order surface source (HS) method for similar airfoils. This was also shown recently by Zedan and Abu-Abdou[5] for Karman-Trefftz airfoils with small trailing-edge angles. Using higher-order surface vortex method, Dutt[9] showed this difficulty can be avoided. However, according to Maskew,[10] one should not get the impression that higher-order methods are always better. For a detailed discussion on the accuracy of the panel methods in relation to the order of singularity distribution over elements, the reader is referred to a paper by Oskam.[11]

The preceding discussion leads us to conclude that, in spite of the popularity of panel methods, there is still a need for a method that could give an exact solution—in the limit—for arbitrary airfoils. To have merit, this method should be simple and numerically efficient. The objective of this paper is to develop such a method. The proposed method utilizes one inverse Joukowski transformation to transform an arbitrary airfoil into a near-circular shape. The flow around this shape is then solved by using a complex-potential series. The performance of the method is evaluated using a number of airfoils that have exact solutions. Results obtained for these airfoils using the HS panel method are also presented to provide a "yardstick" by which the present method could be judged.

General Mathematical Description

The method starts like most conformal mapping methods by transforming the arbitrary airfoil in the z plane (Fig. 1) backward into a near-circular shape in the ζ_1 plane (Fig. 2),

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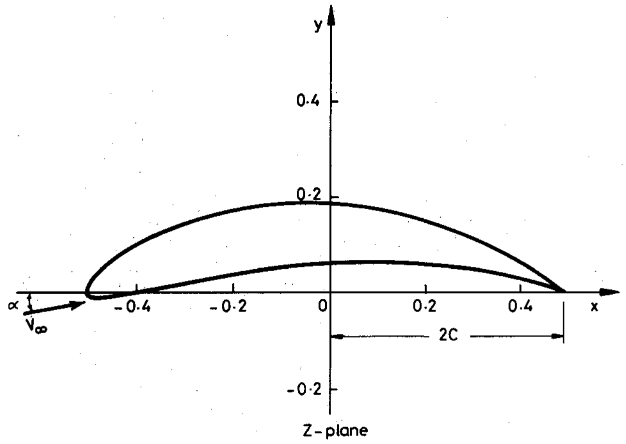

Fig. 1 Airfoil geometry in the physical plane.

using the Joukowski transformation

$$z = \zeta_1 + c^2/\zeta_1 \tag{1}$$

where $z = x + iy$ and $\zeta_1 = \xi_1 + i\eta_1$.

The constant c is estimated as one-fourth of the distance between the trailing edge and a point midway between the leading edge and the center of curvature of the nose (if known) or, using the following approximate relation derived originally for Joulowski airfoils,

$$c/l \simeq 0.25/[1 + (4\varepsilon_x/l)^2] \tag{2a}$$

where

$$\varepsilon_x/l = 0.19245\tau \tag{2b}$$

where τ is the thickness ratio and l the length of the airfoil. The axes in the ζ_1 plane are then shifted to the center of area of the near-circular shape and rotated an angle α, such that the ξ axis is in the direction of freestream (Fig. 2). Let us call the coordinate plane after axes shift and rotation the ζ plane ($\zeta = \xi + i\eta$).

Keeping in mind that $|\mathrm{d}\zeta/\delta\zeta_1| = 1$, the velocity in the airfoil plane is related to the velocity at the corresponding point in the ζ plane by the relation

$$V_z = V_\zeta \left|\frac{\mathrm{d}\zeta_1}{\mathrm{d}z}\right| \tag{3}$$

where

$$\left|\frac{\mathrm{d}\zeta_1}{\mathrm{d}z}\right| = 1\bigg/\left|1 - \frac{c^2}{\zeta_1^2}\right| \tag{4}$$

Thus, if we can determine the velocity distribution on the surface of the near-circular shape in the ζ plane, the determination of the velocity distribution on the airfoil surface becomes straightforward.

Solution of the Flow Past the Near-Circular Shape

The complex potential of the flow past the near-circular shape in the ζ plane may be represented by

$$W(\zeta) = V_\infty \zeta + \sum_1^\infty \frac{c_j}{\zeta^j} + \frac{i\Gamma}{2\pi}\ell n\zeta \tag{5}$$

The first derivative of the preceding expression $\mathrm{d}W/\mathrm{d}\zeta$ (which represents the complexity velocity) is analytic everywhere except at one isolated singular point $\zeta = 0$. This singularity is within the contour. Since we are interested in the solution around the contour, such a singularity poses no trouble. It is interesting to note also that the assumed expression for the complex potential in Eq. (5) satisfies the condition that $\mathrm{d}W/\mathrm{d}\zeta \to V_\infty$ as $\zeta \to \infty$. The first term in this expression is due to the uniform stream; the second is a series containing a doublet at the original and higher-order terms to account for the deviation of shape from an exact circle. The last term represents a clockwise circulation that is necessary to have zero velocity at point "T," which corresponds to the airfoil trailing edge, as required by the Kutta condition.

Splitting both sides of Eq. (5) into real and imaginary and recognizing that $W = \phi + i\psi$ where ϕ is the velocity potential and ψ is the stream function, it can be shown that

$$\psi = V_\infty r \sin\theta + \frac{\Gamma}{2\pi}\ell n r + \sum_{j=1}^{\infty}\left[a_j\left(\frac{-\sin j\theta}{r^j}\right) + b_j\left(\frac{\cos j\theta}{r^j}\right)\right] \tag{6}$$

where $c_j = a_j + ib_j$ and with $\zeta = re^{i\theta}$.

To represent the flow around a cylinder, only the real part of the first term in the preceding series is sufficient. For a near-circular shape, the above series is expected to converge fast. Therefore, only a limited number of terms m need to be retained in the series.

Since the near-circular contour is a streamline, the value of ψ is constant along it, say, at some value ψ_0. With this, Eq. (6) may be written as

$$\sum_1^m a_j\left(\frac{\sin j\theta}{r^j}\right) + \sum_1^m b_j\left(\frac{-\cos j\theta}{r^j}\right) - \Gamma\left(\frac{\ell n r}{2\pi}\right) + \psi_0 = V_\infty r \sin\theta \tag{7}$$

The velocity components u and v are obtained by differentiating the complex potential

$$\frac{\mathrm{d}W}{\mathrm{d}z} = u - iv = V_\infty + \frac{i\Gamma}{2\pi}\frac{1}{\zeta} + \sum_1^m (a_j + ib_j)\left(\frac{-j}{\zeta^{j+1}}\right)$$

Therefore

$$u = V_\infty + \Gamma\left(\frac{\sin\theta}{2\pi r}\right) + \sum_1^m a_j\left[\frac{-j\cos(j+1)\theta}{r^{j+1}}\right] + \sum_1^m b_j\left[\frac{-j\sin(j+1)\theta}{r^{j+1}}\right] \tag{8}$$

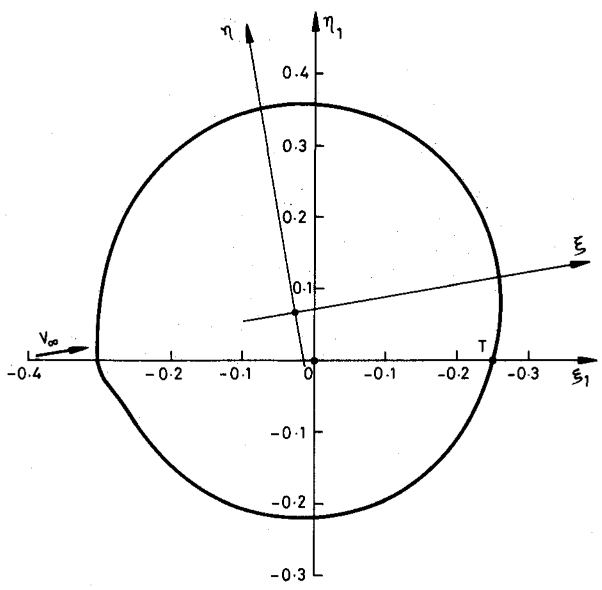

Fig. 2 Near-circular shape in transformed plane.

and

$$v = \Gamma\left(\frac{-\cos\theta}{2\pi r}\right) + \sum_{1}^{m} a_j\left[\frac{-j\sin(j+1)\theta}{r^{j+1}}\right] + \sum_{1}^{m} b_j\left[\frac{j\cos(j+1)\theta}{r^{j+1}}\right] \tag{9}$$

The Kutta condition requires that point "T" be a stagnation point, i.e., $u_T = v_T = 0$. Thus,

$$\sum_{1}^{m} a_j\left[\frac{j\cos(j+1)\theta_T}{r_T^{j+1}}\right] + \sum_{1}^{m} b_j\left[\frac{j\sin(j+1)\theta_T}{r_T^{j+1}}\right] + \Gamma\left(\frac{-\sin\theta_T}{2\pi r_T}\right) = V_\infty \tag{10}$$

$$\sum_{1}^{m} a_j\left[\frac{-j\sin(j+1)\theta_T}{r_T^{j+1}}\right] + \sum_{1}^{m} b_j\left[\frac{j\cos(j+1)\theta_T}{r_T^{j+1}}\right] + \Gamma\left(\frac{-\cos\theta_T}{2\pi r_T}\right) = 0 \tag{11}$$

Determination of Series Coefficients (a_j and b_j and circulation Γ)

With m terms retained in the series complex potential, we have $2m$ unknown coefficients in addition to the circulation Γ and the value of ψ_0. Applying the condition of constant stream function [Eq. (7)] at a number of control points $n = 2m$ on the near-circular contour and the Kutta condition represented by Eqs. (10) and (11) at point T provides the $2m + 2$ equations needed to determine all unknowns. Another possibility is to have an overdetermined system of equations by using a number of control points $n > 2m$. A least-square procedure is then used to determine a solution for this system. It has been found that this alternative procedure is quite helpful when either the contour is not perfectly smooth or when the matrix of the original system of equations is not well-conditioned. This is discussed further in the results section.

Lift Coefficient and Velocity Distribution

The lift force per unit depth is given by the Kutta-Joukowski formula

$$F_L = \rho \Gamma V_\infty \tag{12}$$

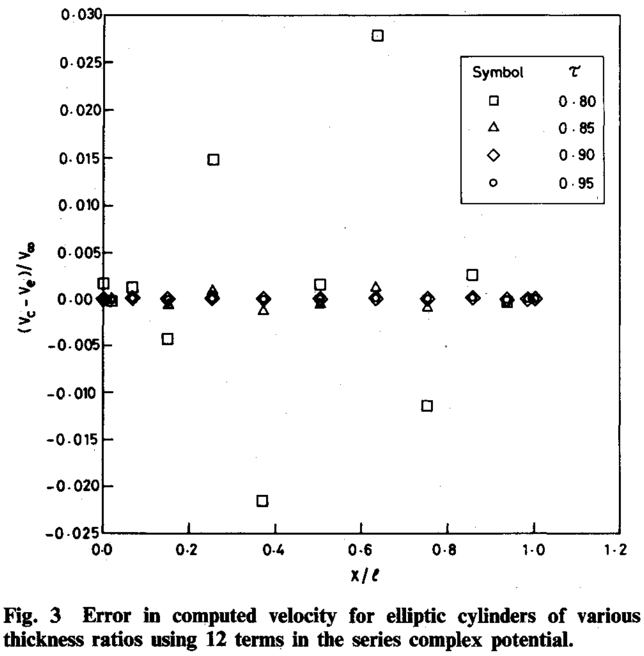

Fig. 3 Error in computed velocity for elliptic cylinders of various thickness ratios using 12 terms in the series complex potential.

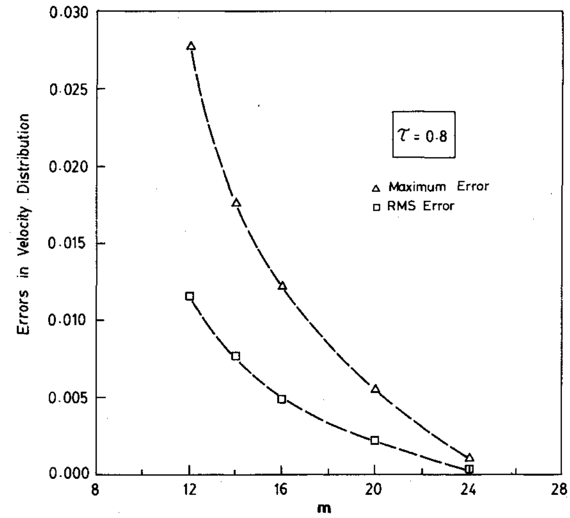

Fig. 4 Root-mean-square and maximum errors in computed velocity distribution of an elliptic cylinder with $\tau = 0.8$.

Thus, the lift coefficient C_L is given by

$$C_L = 2\Gamma / V_\infty l \tag{13}$$

The value of Γ is obtained from the solution of the system of equations, as just described.

The velocity components on the contour are obtained by substituting the coefficients a_j and b_j and the circulation Γ into Eqs. (8) and (9). The total velocity is

$$V_\zeta = \sqrt{u^2 + v^2}$$

The velocity along the airfoil surface V_z is obtained from V_ζ, using Eq. (3).

Verification of the Present Method

The present method to calculate the flow around arbitrary airfoils was verified on two levels. The first and most important is the verification of the series complex-potential solution of the flow around the near-circular shape in the ζ plane. The second level is the investigation of the performance of the method as a whole when applied to airfoils. This performance is compared with the performance of one of the established panel methods due to Hess and Smith.[6]

Verification of the Series Complex-Potential Solution

The solution of the flow around the near-circular shape was checked using the known exact solution of the flow past elliptic cylinders at zero angle of attack. The effect of the number of terms m in the series of Eq. (5) on the accuracy of the solution for different ellipse thickness ratios τ was investigated. With only 12 terms ($m = 12$) and using $n = 24$, the computed normalized velocities agreed with the exact solution up to 6 decimal points for $\tau \geq 0.95$, up to 4 decimal points for $\tau = 0.9$, and up to 3 decimal points for $\tau = 0.85$. For these cases, the exact and computed velocities were almost indistinguishable when plotted on the same graph; therefore, the errors in the velocity distributions are presented instead, as shown in Fig. 3. For $\tau = 0.8$, differences between computed and exact velocities were observed in the second decimal digit. By increasing the number of terms of the series complex potential m and, consequently, the number of control points, the velocity errors for this thickness ratio decreased monotonically to almost nil at $m = 24$ ($n = 48$). This is confirmed by the results of Fig. 4, in which the root-mean-square (rms) of the errors in the velocity distribution, as defined by

$$\text{rms} = \sqrt{(V_{\text{calc}} - V_{\text{exact}})^2/n}$$

is plotted against m. For $\tau \leqslant 0.75$, the errors increased sharply and the method appeared to diverge.

Based on the preceding results, it is recommended that the "thickness" ratio of the near-circular shape in the ζ plane should be greater than 0.8. Fortunately, proper use of the inverse Joukowski transformation for arbitrary airfoils gives contours in the ζ plane that are much closer to a circle than to an ellipse with $\tau = 0.8$. Therefore, this does not present a limitation when application to arbitrary airfoils is the objective.

Application to Airfoils

The method was applied to a number of airfoils of widely varying geometry that have exact solutions. Three Karman-Trefftz (KT) airfoils and the NACA 0012 profile were used. The KT airfoils are obtained by the conformal mapping of a circle, using a special transformation that has a parameter that controls the trailing-edge angle β. Results for KT airfoils with $\beta = 0$ are not presented here since such airfoils are of the Joukowski-type. This type of airfoil can be transformed into an exact circle in the ζ plane and, therefore, presents no challenge for the present method. The "exact" solution for the NACA 0012 is given by Abbott and Doenhoff[2] for an angle of attack α that gives unity lift coefficient. As mentioned, the results obtained by the Hess-Smith panel method are included in the comparisons discussed below. This panel method is based on constant intensity surface source/sink elements. The program given by Moran[12] for this method was used with 100 panels to produce the results presented here.

The effect of the number of terms retained in the series complex potential on the accuracy of the solution for the preceding airfoils was somewhat different from the case of elliptic cylinders. Keeping $n = 2m$ to obtain a closed system of equations, small values of m (< 10) produced results that were smooth but not very accurate. When m was increased, the velocity distribution for some airfoils showed little oscillations around a mean line that increasingly approached the exact solution. It was felt, at this point, that the number of terms in the series should be kept low to avoid round-off errors, especially when using a low-precision machine such as a PC. However, with a low value of m and with $n = 2m$, the number of control points n is not satisfactory to describe faithfully the geometry of the near-circle in the ζ plane. It was decided, therefore, to use a low value of m together with a relatively large value of n ($n > 2m$) resulting in an overdetermined system of linear equations. A standard least-square error minimization scheme was used to solve this system to obtain the series coefficients. Excellent results for the airfoils under consideration were obtained with $m = 12$ and with n in the range 34-48.

Figure 5 shows the velocity distribution for a thick KT airfoil ($\tau = 0.16$) with a small camber ($\varepsilon = 0.0406$) and small trailing-edge angle ($\beta = 4$ deg) at an angle of attack $\alpha = 9$ deg. The figure indicates that the present method is generally more accurate than the HS method, especially near the trailing edge. As for C_L, the present method gave a value of 1.670 and the HS method gave 1.605 whereas the exact value is 1.683. The next test case is another KT airfoil but with large camber ($\varepsilon = 0.123$), small thickness ($\tau = 0.062$), and a small trailing-edge angle ($\beta = 4$ deg) at $\alpha = 9$ deg. The comparison shown in Fig. 6 indicates that whereas both methods predict the velocity distribution accurately in general, the present method predicts the leading-edge suction peak closer to the exact solution. The figure also shows that the accuracy of the present method over the HS method becomes more obvious as the trailing edge is approached. The calculated lift coefficient is 2.665 for the present method and 2.554 for the HS method, compared to an exact value of 2.653. Figure 7 shows the results for the last of the KT airfoils, which is thick and highly cambered and has a large trailing-edge angle ($\tau = 0.136$, $\varepsilon = 0.123$, $\beta = 14.4$ deg). Again, the agreement between calculations and exact solution at $\alpha = 9$ deg is very good everywhere except for small errors near the trailing edge, where the HS method is more accurate. However, the lift coefficient calculated by the present method (2.815) agrees more closely with the exact value (2.847) than that calculated by the HS method (2.782). It should be noted here that the present method results reported above for the KT airfoils were obtained with $m = 12$ and $n = 48$.

The last test case is the symmetric NACA 0012 section whose "exact" pressure distribution is given by Abbot and Doenhoff.[2] This airfoil is widely used as a test case for various

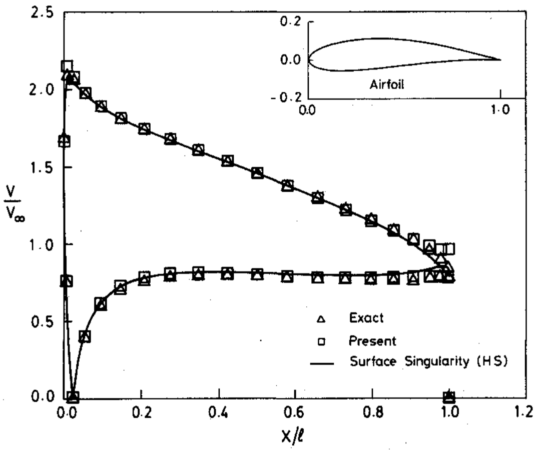

Fig. 5 Velocity results for a thick KT airfoil ($\tau = 0.16$, $\varepsilon = 0.0406$, and $\beta = 4$ deg) at $\alpha = 9$ deg.

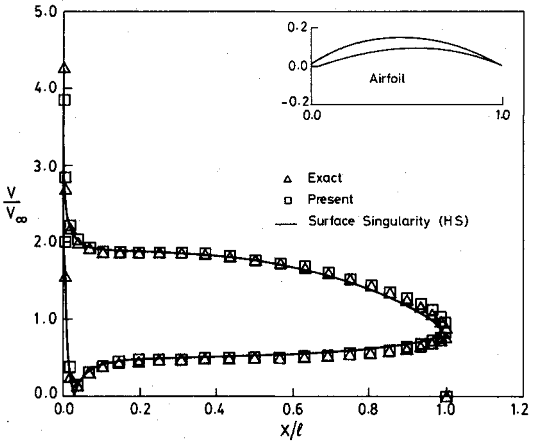

Fig. 6 Velocity results for a highly cambered KT airfoil ($\tau = 0.062$, $\varepsilon = 0.123$, and $\beta = 4$ deg) at $\alpha = 9$ deg.

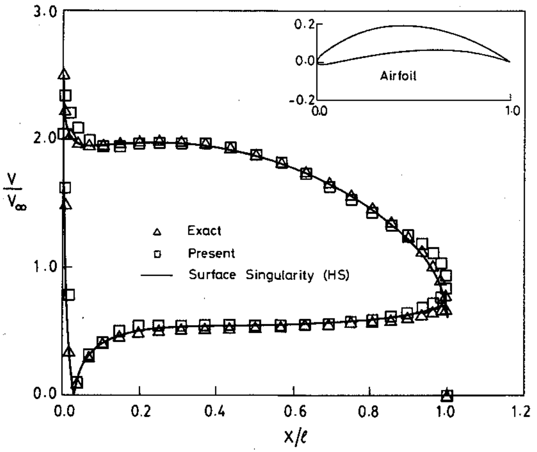

Fig. 7 Velocity results for a thick, highly cambered KT airfoil with relatively large trailing-edge angle ($\tau = 0.136$, $\varepsilon = 0.123$, and $\beta = 14.4$ deg) at $\alpha = 9$ deg.

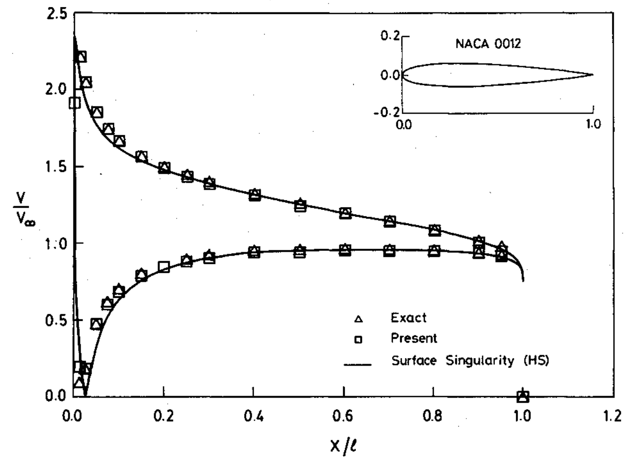

Fig. 8 Velocity results for the NACA 0012 profile at $\alpha = 8.3$ deg.

methods (Anderson,[3] Moran,[12] etc). Anderson gave $\alpha = 9$ deg whereas Moran gave $\alpha = 8.3$ deg corresponding to $C_L = 1.0$ for this airfoil. Using a highly accurate panel method that utilizes elements with linearly varying vortex intensity,[13] the value 8.3 deg appears to be more accurate than 9 deg. Figure 8 shows a comparison of results for $\alpha = 8.3$ deg. The "exact" velocity distribution in the figure was calculated from the pressure distribution given by Abbot and Doenhoff.[2] With $m = 12$ and $n = 34$, the present method clearly outperforms the HS method with 100 panels for this case, especially near the leading edge. The computed lift coefficients were 0.999 (present) and 0.963 (HS). Such values are consistent with the trends in the velocity results of Fig. 8.

Summary and Conclusions

An accurate method has been developed to calculate potential flow around airfoils of arbitrary shapes. The method is both concise and numerically efficient. An inverse Joukowski transformation is used to transform the airfoil contour into a near-circular shape. A scheme based on a series complex potential has been developed to solve the flow around this shape. The general method was verified on two levels. First, the calculation of the flow around the near-circular shape was checked using an ellipse as a test case. Accurate results were obtained for thickness ratios $\tau > 0.8$. Results that are exact to six digits were obtained for $\tau > 0.9$. The second level of verification was for the general method as a whole. Three Karman-Trefftz airfoils covering a wide range of geometrical parameters (ε, τ, and β) and the NACA 0012 airfoil section were used as test cases. Results were compared with both the exact solution and those obtained by the Hess-Smith surface source panel method with 100 panels. The velocity distribution predicted by the present method, with only 12 terms in the complex-potential series, was more accurate than that of the panel method in three test cases and slightly less accurate in one case. The present method was more accurate in predicting the lift coefficient in all cases. Using the same PC machine and compiler, the computation time used by the present method is roughly one-eighth of that of the panel method. However, the panel method still has the advantage of handling multielement airfoils, which the present method cannot handle.

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